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EDUCATIONAL TESTIMONY CONCERNING THE CALCULUS.

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The following notes refer primarily to persons studying the Calculus without the advantage of an experienced instructor.

1. An eminent teacher long familiar with Algebra and Geometry writes : "My efforts to learn the Calculus, for several years, were entirely unsuccessful. I had tried three different authors, without making any progress. Lately I procured a fourth treatise, which has given me quite an insight, so that I begin to enjoy the study. Most of my leisure will now be given to this favorite treatise."

2. Another similar proficient in Algebra and Geometry says : "I procured a work on the Calculus some months ago, intending to master it ; but there is something so obscure in the first part, that I cannot understand it. The rules for differentiating are plain enough. I can differentiate, but I don't understand what it signifies."

3. While writing this, a lady teacher calls to say that her "class in Geometry has been at a stand, for more than a week, over the proposition that a circle is a polygon of an infinite number of sides. Can the contour of a polygon with its straight lines and angles ever be a circle ; or can a circle be a polygon ?" Of course, it is the difficulty of conceiving how the small difference between the inscribed and the circumscribed polygon entirely vanishes.

4. A cadet at West Point, extremely fond of mathematics, thus estimates the Calculus : "The inventors of the Differential and Integral Calculus have claimed that this branch of so-called science belongs to the department of mathematics ; and laboring under that delusion, have introduced it into the course of academical instruction for the torture of

students. Such classification is obviously incorrect, because the principles of mathematics fall within the scope of the reasoning faculty. The Calculus, on the contrary, lies without the boundaries of reason." (Life of Gen. Nathaniel Lyon, p. 30.)

5. Bishop Berkley in 1734: "I have no controversy about your conclusions, but only about your logic; and it must be remembered that I am not concerned about the truth of your theorems, but only about the way of coming at them."

Leaving the collection of further abundant testimony to some educational commission, we find the modern treatises to be, in general, extremely well written. The discouragement and demoralization of learners appear to be caused by two or three pages only, near the beginning, which describe what is termed, "passing to the limit." In this, if our examination be not mistaken, the suppressed premise of the logic consists in a misconception of the quantity usually denoted by $u' - u$. Under previous training, the student naturally regards it as a homogeneous magnitude, whereas the principles of logic require him to regard it here *as a graded series*, like the right hand member of the equation.

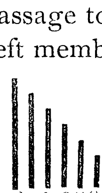
Let the common operation of the student's mind be attentively observed. In resolving quadratic equations, he has been accustomed to regard the similar quantity as a single magnitude, and very naturally he still so regards it; thus, in the ordinary course,

$$u' - u = ah + bh^2, \text{ or}$$
$$\rule{4em}{3pt} = ah + bh^2.$$

And when the equation is divided by h, his idea is still that of a single magnitude in the quotient; thus

$$\frac{\rule{4em}{3pt}}{h} = a + bh.$$

Consequently, the passage to the limit or decreasing h to zero, has some effect upon the left member, to him incomprehensible; thus,



$$000\,000\frac{0}{0} = a + b \times 0.$$

Again and again the learner attempts to form a clear conception of the effect in the left hand member, and more often falters and recoils in the useless effort. Is it not like soldiers repeatedly attempting to storm

an impregnable redoubt, till they become demoralized ? Nor should it be forgotten that ordinary learners have never had the previous training in Wallis's Arithmetic of Infinites and in the subtle metaphysics of the middle ages, which Newton and Leibnitz had enjoyed.

But what is the remedy ? Our answer would be, primarily, to regard the quantity $u' - u$ as a graded series, like the right hand member ; and secondly, to substitute such graded series always in place of $u' - u$, before passing to the limit, or in the language of Newton, "equating the homologous terms of the resulting equation."

"This Calculus," says Laplace, "is only the comparison of the coefficients of the same powers of the differentials or increments, in the development in series, of functions identically equal." (Essai Philosophique sur les Probabilites, p. 56).

In this aspect, the first principles of the Calculus may be correctly derived from the method of indeterminate coefficients. Indeed, some of the simpler applications are already introduced into Algebra under the name of "derived polynomials," or other designations. It were a desirable improvement in works of Algebra, after Indeterminate Coefficients to devote a few pages to the foundation of the Calculus, in connection with the simple notation of Leibnitz, now almost universally acceptable. This would be immediately useful in unfolding the theory of the higher equations, besides other important advantages. For illustration, let the expansion of the binomial $(x + h)$ be denoted by the following notation :

$$(x + h)^n = x^n + nx^{n-1}h + n(n-1)x^{n-2}\frac{h^2}{1.2} + n(n-1)(n-2) \times x^{n-3}\frac{h^3}{1.2.3} + \ldots.$$

$$(x + h)^n = fx + f'x.h + f''x.\frac{h^2}{1.2} + f'''x.\frac{h^3}{1.2.3} + \ldots.$$

$$u' = u + \frac{du}{dx}.h + \frac{d^2u}{dx^2}.\frac{h^2}{1.2} + \frac{d^3u}{dx^3}.\frac{h^3}{1.2.3} + \ldots.$$

Any algebraic expression containing x is termed a function of x, and may be denoted by fx, or by the single letter u. In the foregoing equations, $u = fx = x^n$ is the primitive function. Also $f'x = nx^{n-1}$ is termed the *first* derivative function; $f''x = n(n-1)x^{n-2}$ is termed the *second* derivative function, and so on. The first derivative is more usually denoted by the ratio

$$\frac{du}{dx},$$

where du and dx denote two undetermined increments, named differential of x, differential of u. The second derivative is the ratio of the second differential of u divided by the square of the differential of x; and so on. It should be observed that $f'x$ or

$$\frac{du}{dx},$$

etc., so far correspond entirely to quantities heretofore denoted by A, B, etc., in the common algebraic operations of Indeterminate Coefficients.

This notation is not restricted to the simple case of the binomial just described. Let $u = fx$ denote any function whether a single term or a polynomial, radical or fractional. And let $u' = f(x + h)$ denote its value when x receives the increment or decrement h. Now suppose the function to be expanded by the binomial theorem aided by algebraic division or multiplication, into an ascending series of the powers of h. Since h is an undetermined increment we may suppose it to be so small that the series will be convergent. Denoting the coeffiicients by the same notation as before, we have the augmented function,

$$u' = u + \frac{du}{dx}.h + \frac{d^2u}{dx^2}.\frac{h^2}{1.2} + \frac{d^3u}{dx^3}.\frac{h^3}{1.2.3} + \ldots.$$

This admirable extension of the binomial formula is called from Dr. Brook Taylor, its author, Taylor's Theorem of increments. So far as the coefficients of the powers of h denote mere quantity, no further demonstration is needed ; but they were intended to denote their mode of derivation also. Here now is the graded series for establishing the Calculus with all the rigorous accuracy of Algebra. For example, let $u = ax^2 + bx + c$.

Let u' denote the augmented value when x received the undetermined increment h.

$$u' = ax^2 + bx + c + (2ax + b)h + ah^2,$$
$$u' - u = (2ax + b)h + ah^2.$$

Substituting in place of $u' - u$ the series of Taylor's Theorem,

$$\frac{du}{dx}.h + \frac{d^2u}{dx^2}.\frac{h^2}{2} + \ldots. = (2ax + b)h + ah^2.$$

Or dividing through by h,

$$\frac{du}{dx} + \frac{d^2u}{dx^2} \cdot \frac{h}{2} + \ldots = 2ax + b + ah.$$

"Passing to the limit," that is making h to be 0,

$$\frac{du}{dx} = 2ax + b.$$

In this manner, the usual difficulty is entirely avoided. In indicating this modification, however, we have followed the popular notion that learners of the Calculus must first attend to some characteristic demonstration. But really, on reconsidering the question, what does it amount to? We have denoted the indeterminate coefficient of h by the symbol

$$\frac{du}{dx},$$

have gone through the usual course of demonstration, and – reproduced the original definition. "Passing to the limit" has amounted to nothing more.

The important inference hence arises, that a mere definition of the derivative

$$\frac{du}{dx},$$

as the coefficient of h, will suffice at the beginning, without that most unfortunate initiation of "passing to the limit." We can therefore commend to writers of Algebra the insertion of the formal notation and first simple rules of the Calculus, as an extension of the method of indeterminate coefficients, with still more satisfaction and confidence.

One further remark,—most writers have preferred the "infinitesimals" of Leibnitz to the "flowing quantities" of Newton. But at the first, leaving these to future applications, will it not be simpler to regard the quantities as primitive and derivative only, like the binomial coefficients or terms? As some ingenious writer has remarked, the increments need only be so small, that to suppose them smaller would not change the character of the results.

INTEGRATION OF POLYNOMIAL DIFFERENTIALS GENERAL FORMULAE OF REDUCTION.

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Reduction Formulae, whereby the integration of a given differential is made to depend on that of another of a more simple form, play an important part in the Integral Calculus. They are usually obtained by the